M-math 1st year Mid Semester Exam
Subject : Advanced Probability

Time : 3 hours Date : 10-9-14. Max.Marks 50.

1. Let X and Y be real valued r.v.'s defined on a probability space $(\Omega, \mathcal{F}, P)$. Determine (explicitly) a regular conditional distribution of X given Y in each of the following cases .
a) X and Y have joint density $f(x, y)$.
b) $X : \Omega \to \{0,1\}^n$ and $Y : \Omega \to [0,1]$ having the joint distribution

$$P\{X = x, Y \in B\} = \int_B y^k (1-y)^{n-k}\, dy$$

where $x = (x_1, \cdots, x_n) \in \{0,1\}^n, k :=$ Cardinality of $\{\mathrm{i} : \mathrm{x_i} = 1\}$ and $B \subset [0,1]$, a Borel set. (5+5)

2. a)Let $\{Y_n; n \geq 1\}$ be an independent sequence with $EY_1^2 < \infty$ and $\mathcal{F}_n := \sigma\{Y_i : 1 \leq i \leq n\}, n \geq 1$ and $\mathcal{F}_0 = \{\phi, \Omega\}$. In each of the following cases show that $\{X_n, \mathcal{F}_n, n \geq 0\}$, is a square integrable martingale and compute the predictable increasing process $\{< X >_n, n \geq 0\}$ such that $\{X_n^2 - < X >_n, n \geq 0\}$ is an $(\mathcal{F}_n)$ martingale.
i) $X_n = Y_1 + \cdots + Y_n, n \geq 1$ and $X_0 = 0 = EY_i, i \geq 1$.
ii) $X_n = \Pi_{i=1}^n Y_i, n \geq 1$ and $X_0 = 1 = EY_i, i \geq 1$.
b) Show that if $\{X_n, \mathcal{F}_n, n \geq 0\}$ is a predictable martingale with $X_0 = 0$ $a.s.$ then $X_n = 0$ $a.s.$ for all $n \geq 0$. (5+5)

3. Let $\{\mu_n\}$ be a sequence of probability measures on $\mathbb{R}$. In each of the following cases , determine if $\{\mu_n\}$ converges weakly . Prove your result.
a) $\mu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{\frac{i}{n}}$, where δ_x is the Dirac measure at x.
b) $\mu_n := \lambda|_{[-n,n]}$, the restriction of the Lebesgue measure λ to the interval $[-n, n]$.
c) $\mu_n := P_{\frac{X_n}{n}}$, the distribution of the r.v $\frac{X_n}{n}$ where X_n has the Geometric distribution with parameter $\frac{1}{n^2}$. (4+3+3)

4. Let μ_n, μ be probability measures on $\mathbb{R}^k$ and let $\hat{\mu_n}, \hat{\mu}$ denote their characteristic functions.

a) Show that $\hat{\mu}$ is uniformly continuous on $\mathbb{R}^k$.
b) Suppose that $\hat{\mu_n}(t) \to \hat{\mu}(t)$ for every $t \in \mathbb{R}^k$. Show that the family $\{\mu_n\}$ is tight. (4+6)

5. Let F, G be (probability)distribution functions on $\mathbb{R}$ and define

$$d(F,G) := \{\epsilon : G(x-\epsilon) - \epsilon \leq F(x) \leq G(x+\epsilon) + \epsilon, \forall x \in \mathbb{R}.\}$$

Show that
a) $d(.,.)$ is a metric and
b) $d(F_n, F) \to 0$ iff $F_n(x) \to F(x)$ for all continuity points x of F. (7+8)